THE STABILITY OF PLANE POISEUILLE FLOW IN THE PRESENCE OF ELASTIC BOUNDARIES

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An investigation of the stability of Poiseuille flow in a channel with compliant walls is conducted on the basis of the method of small oscillations [1 to 4].

Because of the approximate nature of the method of finding solutions of the Orr-Sommerfeld equation [5], various calculation formulas for the critical Reynolds number can be obtained depending on the degree of approximation. Therefore, in the present paper a scheme for calculating the critical Reynolds number for Poiseuille flow between rigid walls is presented, which is then generalized to the case of elastic boundaries. The formulation of the boundary conditions for the perturbations on the compliant surface differs from the corresponding formulations contained in papers [6 and 11].

1. Stability of Poiseuille flow between rigid walls. We shall consider the stability of Poiseuille flow with respect to perturbations of the amplitude of the stream function, which is an even function in the system of coordinates with origin on the axis of the channel [3]. The question of the stability of Poiseuille flow reduces to finding the general solution of Equation

$$(u-c)(f''-\alpha^2 f)-u''f=-\frac{i}{\alpha R}(f^{\mathrm{IV}}-2\alpha^2 f''+\alpha^4 f) \qquad (1.1)$$

for the boundary conditions

$$f(0)=f'(0)=f'(1)=f'''(1)=0 \qquad (1.2)$$

Here $u(y)$ is the velocity distribution in Poiseuille flow; $f(y)$ is the amplitude of the stream function of the perturbing motion; α is the wave number, determined by the wave length of the perturbed motion; c is the velocity of propagation of the perturbing motion; and R is the Reynolds number formed with the half-width of the channel h and the maximum velocity.

All of the quantities which appear in (1.1) and (1.2) are dimensionless. The maximum velocity in the channel is taken as the velocity scale and the half-width of the channel as the length scale. For the existence of a nontrivial solution of Equation (1.1) with boundary conditions (1.2) it is necessary and sufficient that

$$\begin{vmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ f_1'(0) & f_2'(0) & f_3'(0) & f_4'(0) \\ f_1'(1) & f_2'(1) & f_3'(1) & f_4'(1) \\ f_1'''(1) & f_2'''(1) & f_3'''(1) & f_4'''(1) \end{vmatrix} = 0 \qquad (1.3)$$

Here $f_i \ (i = 1, 2, 3, 4)$ are particular solutions of Equation (1.1) [3]. After estimating the individual terms of Equation (1.3), it can be transformed into

$$-\frac{1}{y_k}\frac{f_3(0)}{f_3'(0)}(1+\Delta) = \frac{cu_0'f_2'(1)}{u_0'cf_2'(1)+f_1'(1)} \qquad \left(\Delta = \frac{u_0'y_k}{c} - 1\right) \qquad (1.4)$$

Here $u_0' = u'(0)$ and $u'(y_k) = u_k = c$. The correction term Δ in the left-hand side of (1.4) can be taken equal to zero in the first approximation. Its numerical value for the second approximation is determined from calculations of the first approximation [5]. In the following we shall restrict ourselves to the first approximation, i.e. we shall set $\Delta = 0$. Expressing the left-hand side of (1.4) by means of the Tietjens function [12] we shall substitute the values of $f_1'(1)$ and $f_2'(1)$; we obtain

$$F^\circ(w) = \frac{1}{1-F(w)} =$$

$$= 1 + u_0'c\left(\int_0^1 (u-c)^2 dy\right)^{-1}\int_0^1 (u-c)^2 dy \int_0^y (u-c)^{-2} dy + u_0'c\left(\alpha^2\int_0^1 (u-c)^2 dy\right)^{-1} + O(\alpha^2) \qquad (1.5)$$

In the complex equation (1.5) the values of the Tietjens function $F(w)$ and the Lin function $F^\circ(w)$ do not depend on the velocity distribution $u(y)$. Extensive tables of these functions are contained in [13]. All of the terms of order α^2 and higher are combined in the term $O(\alpha^2)$ in the right-hand side of (1.5). The argument w is defined by

$$w = y_k(u_k'\alpha R)^{1/3} \qquad (1.6)$$

After determining the expressions for the integrals which appear in (1.5), taking into account the Poiseuille flow velocity distribution $u = 2y - y^2$, we obtain the two real equalities

$$F_r^\circ(w) = -\frac{4c\ln c}{u_k'^3} + \frac{1}{\alpha^2}\frac{2c}{8/15 - 4/3c + c^2}, \qquad F_i^\circ(w) = \frac{4\pi c}{u_k'^3} \qquad (1.7)$$

The equalities (1.6) and (1.7) permit the calculations for the neutral stability curve to be carried out in the following order.

1. We are given a value $0 \leqslant F_i^\circ \leqslant 0.58$. From graph (Fig.1) $M(c) = 4\pi c/u_k'^3$ and with the help of the relation $F_i^\circ = F_i^\circ(w)$ we find the values of c and w.
2. From the c which has been found we determine y_k and u_k'.
3. For the c which has been obtained we find F_r°.
4. Using the first equality of (1.7), we find the parameter α.
5. With the help of (1.6) we determine the corresponding value of R.
6. We compute

$$\alpha^* = \alpha\frac{\delta^*}{h} = \frac{1}{3}\alpha, \quad R^* = \frac{1}{3}R \qquad \left(\frac{\delta^*}{h} = \int_0^1 (1-u)\,dy\right)$$

In Fig.2 the neutral curve 1 calculated for Poiseuille flow according to the scheme outlined above is presented along with the curve 2 obtained by Lin [5]. The critical Reynolds number for curve 1 equals $R_*^* = 2120$. For comparison we give the values of the critical Reynolds number calculated by Lin $R_*^* = 1970$ (first approximation) and $R_*^* = 1780$ (second approximation) and by Meksyn $R_*^* = 2260$ [14]. The first approximation of Lin corresonds to the case $\Delta = 0$ considered above. The second approximation of Lin takes the correction term Δ into account. We shall now turn to the case for which the perturbations arising in Poiseuille flow have the amplitude of a stream function which is an odd function with respect to the axis of the channel.

The boundary conditions for Equation (1.1) in this case are written as [3] .

$$f(0) = f'(0) = f(1) = f''(1) = 0 \tag{1.8}$$

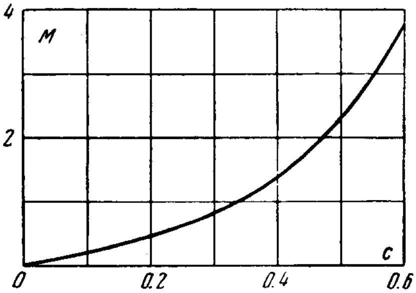

Fig. 1

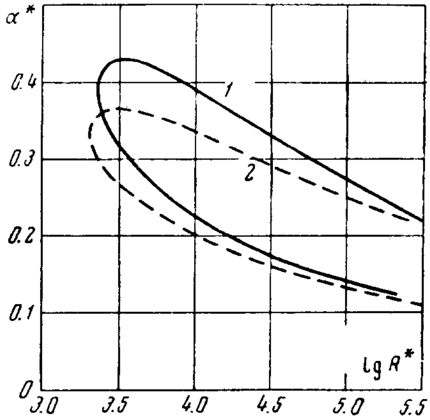

Fig. 2

The characteristic equation of the problem corresponding to the conditions (1.8) has the form

$$\begin{vmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ f_1'(0) & f_2'(0) & f_3'(0) & f_4'(0) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) \\ f_1''(1) & f_2''(1) & f_3''(1) & f_4''(1) \end{vmatrix} = 0 \tag{1.9}$$

After carrying out transformations of the determinant (1.9), similar to those made in the case of the even stream function described above, we obtain

$$F^\circ(w) = 1 + u_0'c\int_0^1 (u-c)^{-2}\,dy + u_0'c\alpha^2\left[\int_0^1 (u-c)^{-2}\,dy \int_0^y (u-c)^2\,dy \int_0^y (u-c)^{-2}\,dy - \right.$$
$$\left. - \int_0^1 (u-c)^{-2}\,dy \int_0^1 (u-c)^{-2}\,dy \int_0^y (u-c)^2\,dy\right] + O(\alpha^4) \tag{1.10}$$

We shall equate the real and imaginary parts of (1.10), after substituting the expressions for the integrals; we obtain

$$F_r^\circ(w) = \tag{1.11}$$
$$= 1 - \frac{u_0'}{u_k'} + \frac{u_0'cu_k''}{u_k'^3}\ln c + \frac{u_0'c}{u_k'^2}\alpha^2\left[-\frac{1}{3} + \frac{7}{36}\frac{u_k''}{u_k'} + \frac{25}{432}\left(\frac{u_k''}{u_k'}\right)^2 - \frac{1}{32}\left(\frac{u_k''}{u_k'}\right)^3\right]$$

$$F_i^\circ(w) = -\pi\frac{u_0'cu_k''}{u_k'^3} \tag{1.12}$$

Both of the conditions (1.11) and (1.12) must be satisfied on the neutral stability curve. The range of variation of w in which the relation (1.12) can be fulfilled, is defined by the inequality $w \geqslant 2.3$, since $F_i^\circ(w) \geqslant 0$. With the variation of w in the region $w \geqslant 2.3$ the following estimates are valid for a Poiseuille profile:

$$c \leqslant 0.246, \quad \frac{u_0'}{u_k'} \leqslant 1.151, \quad \frac{u_0' c u_k''}{u_k'^3} \ln c \leqslant 0.263, \qquad -0.195 \geqslant \frac{7}{36} \frac{u_k''}{u_k'} \geqslant -0.224$$

$$\frac{25}{432} \left(\frac{u_k''}{u_k'}\right)^2 \leqslant 0.0766, \quad -0.0313 \geqslant \frac{1}{32} \left(\frac{u_k''}{u_k'}\right)^3 \geqslant 0.0473, \quad F_r^\circ(w) > 0.96$$

Thus, from (1.11) it is seen that $\alpha^2 > 0$ when the equality (1.12) is fulfilled. This is indicative of the stability of laminar Poiseuille flow to all small perturbations of the form $f(y) \exp[i\alpha(x - ct)]$, where $f(y)$ is an odd function with respect to the axis of the channel.

The result obtained can be illustrated by the following reasons. In the case of $f(y)$ even with respect to the axis of the channel, the velocity of the perturbing motion $v_x(y)$ will be an odd function and, consequently, it will always satisfy the equality

$$\int_0^2 v_x(y)\, dy = 0 \tag{1.13}$$

which expresses the continuity condition since at a given instant exactly the same quantity of fluid flows through each section of the channel. In the case of $f(y)$ odd with respect to the axis of the channel, the velocity component of the perturbing motion v_x will be an even function and, consequently, the integral on the left-hand side of (1.13) need not be equal to zero. In view of the continuity equation this circumstance indicates that the existence of such perturbations is impossible. Without even resorting to detailed analysis, similar arguments permit the conclusion that not all perturbations of stream functions which are odd with respect to the axis of the channel can exist in Poiseuille flow.

2. Boundary conditions for the Orr-Sommerfeld equation in the case of an elastic surface. The problem of the stability of the laminar form of flow near an elastic surface differs in the boundary conditions at $y = 0$ from the analogous problem for a rigid wall.

If in the case of a rigid surface the velocity components of the perturbing motion are equal to zero at the wall, it is then in the case of an elastic surface natural to assume that, by virtue of satisfying the no-slip condition, they are equal to the corresponding velocity components of the points on the surface of the wall.

The deformation of the elastic surface is, in turn, related to the tangential and normal stresses on the surface of the body which are caused by the pulsations of the velocity of the perturbing motion in the flow.

We shall consider only small deformations of the wall so that their influence on the basic velocity profile can be neglected.

We are given the following coupling relations between the stresses and the deformations of the surface:

$$y_1 = k_1 p_1 e^{i\theta_1}, \qquad x_1 = m_1 \tau_1 e^{i\theta_2} \qquad \left(\tau_1 = \mu \frac{\partial v_x}{\partial y_1}\right) \tag{2.1}$$

Here k_1 and m_1 are constant quantities which depend on the properties of the coating, p_1 is the varying pressure component on the surface of the body, τ_1 is the varying tangential stress component on the surface of the body, and θ_1 and θ_2 are the phase shifts between the oscillations of the stresses on the surface of the body and the corresponding deformations.

The pressure pulsations on the wall can be determined from the linearized Navier-Stokes equations for the perturbing motion [7].

Differentiating the equalities (2.1) with respect to time and expressing the varying quantities which appear in them by means of the stream function of the perturbing motion, we obtain the following boundary conditions on the elastic surface:

a) wall compliant only in normal direction

$$-\frac{kc}{i\alpha R} f'''(0) + f(0)\,[e^{-i\theta_1} - kcu_0'] = 0, \qquad f'(0) = 0 \tag{2.2}$$

b) wall compliant only in tangential direction

$$f(0) = 0, \qquad f'(0) + i\,\frac{m\alpha c}{R}\, f''(0)\, e^{i\theta_2} = 0 \tag{2.3}$$

In equalities (2.2) and (2.3) the dimensionless quantities

$$k = k_1 \frac{\rho U^2}{h}, \qquad m = m_1 \frac{\rho U^2}{h}$$

play the role of similarity criteria which characterize the properties of the coating. For exactly the same mechanical characteristics of coating the effectiveness of its performance depends on the density of the fluid flowing in the channel, the surface of the maximum velocity and the width of the channel.

The boundary conditions on the wall thus obtained are a consequence of the assumed relation between the deformations of the surface and the corresponding stresses (2.1) which, in spite of its simplicity, is very general. Thus, for example, the first relation of (2.1) includes all coatings whose deformation is described by an equation of the type

$$L\left(y,\ \frac{\partial y}{\partial t},\ \frac{\partial y}{\partial x},\ \frac{\partial^2 y}{\partial t^2},\ \frac{\partial^2 y}{\partial x^2}, \ldots, \frac{\partial^n y}{\partial x^n}\right) = p(x, t)$$

where L is a linear combination of its arguments with constant coefficients, p is the pressure of the perturbing motion on the surface of the wall, and y is the coordinate of the surface of the elastic coating. It is assumed, as is usual in the method of small oscillations, that all quantities associated with the perturbing motion vary in accordance with the harmonic law

$$A(x, y, t) = [A_r(y) + iA_i(y)] \exp[i\alpha(x - ct)]$$

3. Stability of Poiseuille flow on a surface compliant in the normal direction. Using the boundary conditions (2.2), the following characteristic equation can be obtained:

$$-\frac{1}{y_k}\frac{f_3(0)}{f_3'(0)} - \frac{u_0' kc}{e^{-i\theta_1} - kcu_0'} = \frac{cu_0 f_2'(1)}{u_0' c f_2'(1) + f_1'(1)} \tag{3.1}$$

It is a generalization of relation (1.4) to the case of an elastic wall. In the derivation of (3.1) only terms higher with respect to the quantity αR, which is assumed to be sufficiently large, were taken into account. On the basis of (3.1) we find the two real equations

$$N_r = G_r(w, kc, \theta_1, u_0'), \qquad N_i = G_i(w, kc, \theta_1, u_0') \tag{3.2}$$

relating the characteristics of the perturbing motion α and c, the Reynolds number R, the velocity profile data and the parameters of the elastic coating k and θ_1.

The functions which appear in (3.2) have the following expressions:

$$N_r = -\frac{4c \ln c}{u_k'^3} + \frac{1}{\alpha^2}\frac{2c}{8/15 - 4/3c + c^2}, \qquad N_i = \frac{4\pi c}{u_k'^3}$$

$$G_r = \frac{A - F_r(w)}{(A - F_r)^2 + (B - F_i)^2}, \qquad G_i = \frac{F_i(w) - B}{(A - F_r)^2 + (B - F_i)^2} \tag{3.3}$$

$$A = \frac{1 - kcu_0' \cos\theta_1}{1 - 2kcu_0' \cos\theta_1 + k^2c^2u_0'^2}, \qquad B = \frac{kcu_0' \sin\theta_1}{1 - 2kcu_0' \cos\theta_1 + k^2c^2u_0'^2} \tag{3.4}$$

Using the system of equations (3.2), the following order of calculating the critical Reynolds number for Poiseuille flow between elastic walls can be proposed.

1. Having the numerical values of the parameters kc and θ_1 and knowing the quantity $u_0'' = 2$ (for Poiseuille flow), we compute A and B from Formulas (3.4).

2. Using the known relations $F_i(w)$ and $F_r(w)$ [13 and 15], we construct with the help of Formula (3.3) a graph of the function $G_i(w)$ for the A and B which have been found.

3. From the graph of $G_i = G_i(w)$ we determine G_i ∸ and the corresponding value of w_{max}.

4. After graphically solving the second equation of (3.2) for c, we determine the quantity c_* and, consequently, the corresponding values of y_k and u_k.

5. From Formula (3.3) we determine G_{r*} for the value $w = w_*$.

6. Using the first equality of (3.2), we compute the parameter α.

7. With the help of the relation (1.6), we determine the critical Reynolds number.

8. We find the value of the parameter $'k = kc / c_m$ which corresponds to this Reynolds number.

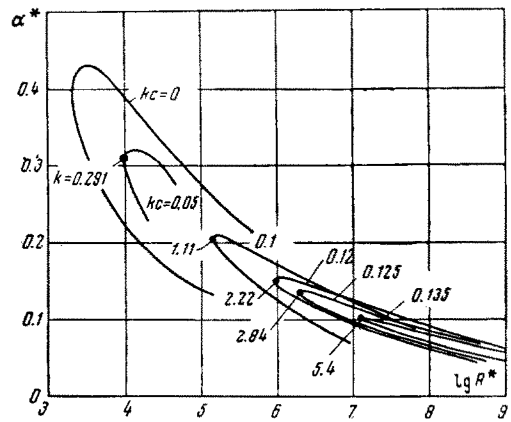

Fig. 3

Performing the indicated calculations for different values of the product kc with constant θ_1, the relation

$$R_*^* = R_*^* (k, \theta_1 = \text{const})$$

can be obtained.

On Fig.3 curves of neutral stability for a Poiseuille profile are presented which were constructed for the case $\theta_1 = 60°$ according to the scheme advanced above.

The calculations indicate that the compliance of the surface in the normal direction can both inctrease the stability of the laminar form of flow ($\theta_1 \approx 60 - 90°$), and decrease it ($\theta_1 \approx 150 - 180°$).

4. Stability of Poiseuille flow on a surface compliant in the tangential plane. Using the homogeneous boundary conditions on the wall (2.3) and the conditions on the axis of the channel from (1.2), the following characteristic equation can be obtained taking into account only terms higher with respect to the quantity αR :

$$\{1 - F(w)[1 - m\alpha^2c^2y_k w^{-3/2} \exp(i(\theta_2 + {}^1/_4\pi))]\}^{-1} = N_r + iN_i \qquad (4.1)$$

The quantities $N_r, N_i, F(w)$ are defined by expressions analogous to those in the equalities (3.2).

The complex equation (4.1) can be replaced by the two real equations

$$\Phi_r = N_r, \qquad \Phi_i = N_i \qquad \left(\Phi_r = \frac{D_r(w, M, \theta_2)}{D_r^2 + D_i^2}, \ \Phi_i = -\frac{D_i(w, M, \theta_2)}{D_r^2 + D_i^2}\right) \qquad (4.2)$$

Here

$$D_r(w, M, \theta_2) = 1 - F_r(w) + F_r(w) M w^{-3/2} \cos(\theta_2 + {}^1/_4\pi) - F_i(w) M w^{-3/2} \sin(\theta_2 + {}^1/_4\pi)$$

$$D_i(w, M, \theta_2) = -F_i(w) + F_i(w) M w^{-3/2} \cos(\theta_2 + {}^1/_4\pi) + F_r(w) M w^{-3/2} \sin(\theta_2 + {}^1/_4\pi)$$

$$M = m\alpha^2c^2y_k$$

Using (4.2) and adhering to the scheme proposed in Section 3, it is not difficult to calculate the critical Reynolds number and the corresponding value of the parameter m.

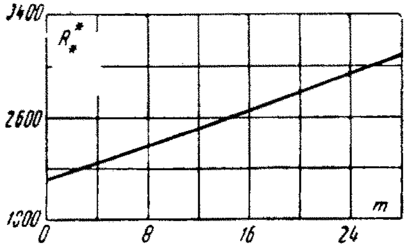

Fig. 4

The results of the calculations of the critical Reynolds number of Poiseuille flow between the flexible walls compliant in the tangential plane are presented in Fig.4. The calculations were made for

Comparison of the stability characteristics of Poiseuille flow between flexible walls compliant in the normal direction and flexible walls compliant in the tangential direction for optimum phase shifts θ_1 and θ_2, leads to the following conclusion. The capacity of the wall to deform in a direction normal to the surface under the action of normal pressure pulsations infuences the stability of the laminar form of flow in significantly greater degree than the corresonding capacity of the wall to deform in the tangential plane. The explanation for this circumstance lies in the fact that the influence of the compliance of the surface in the tangential plane on the stability of the laminar form of flow is inversely proportional to the Reynolds number, which follows from the second boundary condition of (2.3).

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